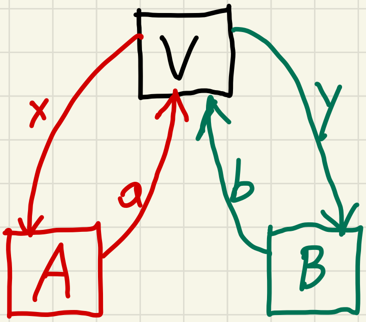
Quantum Morphisms

Lecture 2

Nonlocal Games

Clauser, Horne, Shimony, Holt (CHSH) game



$x, y, a, b \in \{0,1\}$

- Verifier/referee sends Alice + Bob bits x and y respectively.
- Alice + Bob respond with bits a and b respectively.
- They win if $x \wedge y = a \oplus b$.
- They CANNOT communicate during the game.

Example rounds:

	A	B
input	0	1
output	1	1

$x \wedge y = 0 \wedge 1 = 0$

$a \oplus b = 1 \oplus 1 = 0$

Win!

A + B respond same unless $x = y = 1$

	A	B
input	1	1
output	1	1

$x \wedge y = 1 \wedge 1 = 1$

$a \oplus b = 1 \oplus 1 = 0$

Lose :(

Nonlocal Games

Formal definition

A nonlocal game is a tuple $(I_A, I_B, O_A, O_B, \pi, \lambda)$

- I_A, I_B, O_A, O_B – finite sets
- $\pi : I_A \times I_B \to [0,1]$ – a probability distribution
- $\lambda : O_A \times O_B \times I_A \times I_B \longrightarrow \{0,1\}$ – the verification function

$$\lambda(a,b|x,y) = \begin{cases} 1 & \text{win} \\ 0 & \text{lose} \end{cases}$$

CHSH Game: $I_A = I_B = O_A = O_B = \{0,1\}$

$\pi(x,y) = 1/4$ (uniform)

$$\lambda(a,b|x,y) = \begin{cases} 1 & \text{if } x \wedge y = a \oplus b \\ 0 & \text{otherwise} \end{cases}$$

Classical Strategies

Deterministic strategy: a pair of functions $f_A: I_A \to O_A$ + $f_B: I_B \to O_B$

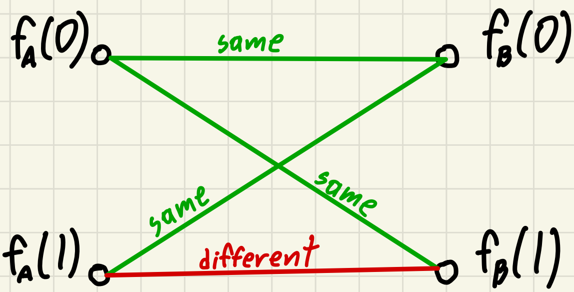
If Alice receives x, she answers $f_A(x)$.
Similarly for Bob.

Example on CHSH: $f_A(x) = f_B(y) = 0 \quad \forall x, y \in \{0,1\}$

the "always answer 0" strategy

$a \oplus b = 0 \oplus 0 = 0 = x \wedge y$ unless $x = y = 1$

Win 3/4 of time



Correlations

$p(a,b|x,y)$:= probability Alice + Bob respond with a + b given they received x + y respectively

Determistic case: $p(a,b|x,y) = \begin{cases} 1 & \text{if } f_A(x) = a + f_B(y) = b \\ 0 & \text{o.w.} \end{cases}$

A classical probabilistic strategy consists of:

- a random variable Z taking values in a set S.
- a pair of functions $f_A : I_A \times S \to O_A$, $f_B : I_B \times S \to O_B$

Alice & Bob "share" Z.

If Z takes value $s \in S$, then A & B play with deterministic strategy $f_A(\cdot, s)$, $f_B(\cdot, s)$

Corresponding correlation:

$$p = \sum_{s \in S} P(Z=s)\, p_s$$

$$p_s(a,b|x,y) = \begin{cases} 1 & \text{if } f_A(x,s)=a \ \& \ f_B(y,s)=b \\ 0 & \text{o.w.} \end{cases}$$

$C_{loc}(O_A, O_B, I_A, I_B) :=$ set of classical correlations

$=$ convex hull of classical deterministic correlations

<u>Value of a game</u>

$$w(G) := \cancel{\sup}\ \max_{p \in C_{loc}} \sum_{\substack{x \in I_A, y \in I_B \\ a \in O_A, b \in O_B}} \pi(x,y)\, p(a,b|x,y)\, \lambda(a,b|x,y)$$

$(O_A, O_B, I_A, I_B, \pi, \lambda)$

Quantum Strategies

A quantum strategy consists of:

- a shared state $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ mixed states?
- a POVM $\mathcal{E}_x = \{E_{xa} \in \mathbb{C}^{d_A \times d_A} : a \in O_A\}$ for each $x \in I_A$
- a POVM $\mathcal{F}_y = \{F_{yb} \in \mathbb{C}^{d_B \times d_B} : b \in O_B\}$ for each $y \in I_B$

$E_{xa} \geq 0$, $\sum_a E_{xa} = I$

Corresponding correlation: $p(a,b|x,y) = \langle \psi | (E_{xa} \otimes F_{yb}) | \psi \rangle$

$C_q(O_A, O_B, I_A, I_B) :=$ set of quantum correlations (this is convex) not closed

Quantum value: $\omega^*(G) := \sup_{p \in C_q} \sum_{x,y,a,b} \pi(x,y)\, p(a,b|x,y)\, \lambda(a,b|x,y)$

Example for CHSH: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$

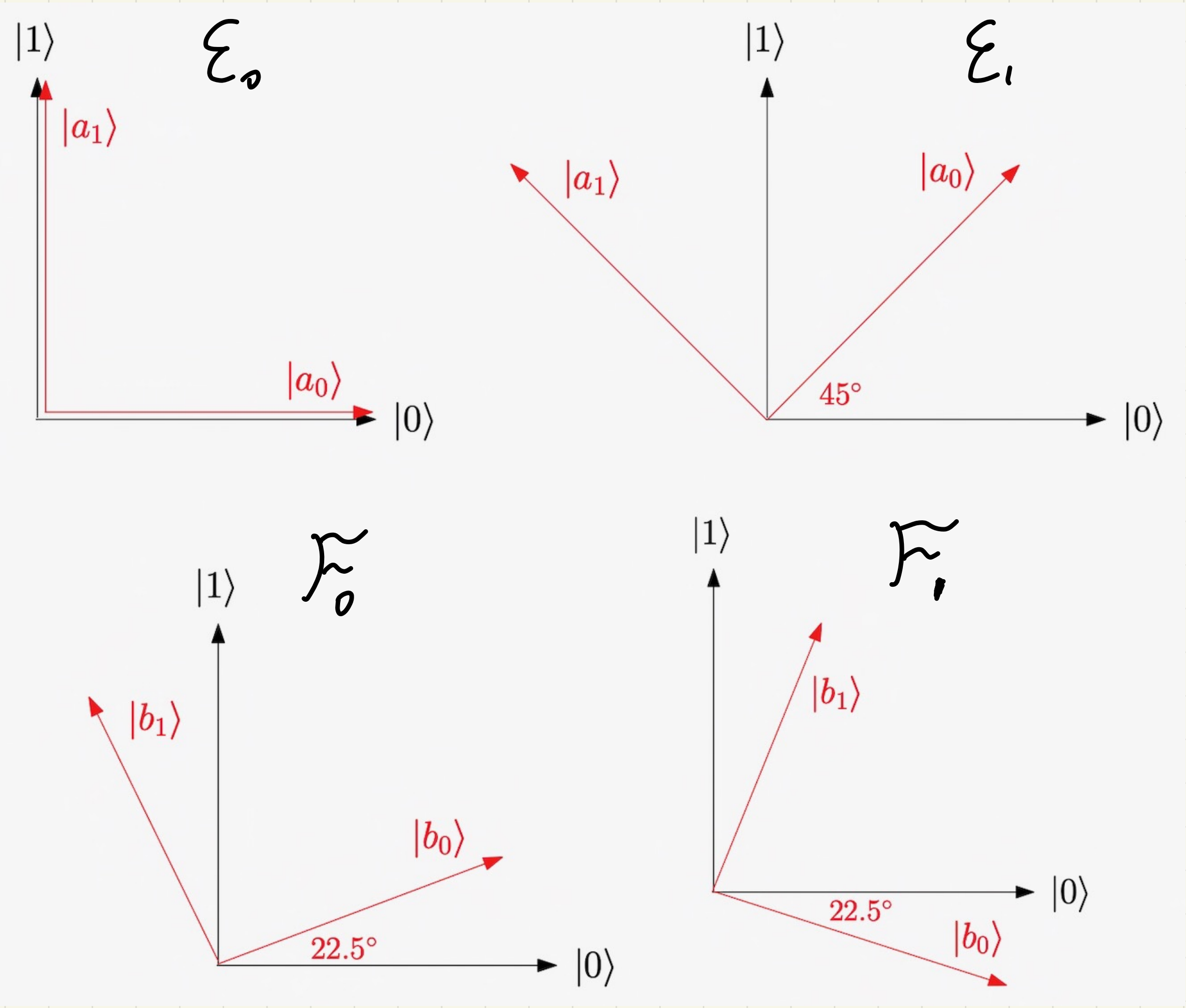
$\mathcal{E}_0 = (|0\rangle, |1\rangle)$ $\quad \mathcal{F}_0 = (\cos\pi/8\,|0\rangle + \sin\pi/8\,|1\rangle, -\sin\pi/8\,|0\rangle + \cos\pi/8\,|1\rangle)$

$\mathcal{E}_1 = (|+\rangle, |-\rangle)$ $\quad \mathcal{F}_1 = (\cos{-\pi/8}\,|0\rangle + \sin{-\pi/8}\,|1\rangle, -\sin{-\pi/8}\,|0\rangle + \cos{-\pi/8}\,|1\rangle)$

$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

Exercise: show that this strategy wins the CHSH game w/ probability $\cos^2(\pi/8) \approx 0.854$

Intuition for the strategy



The non-signalling property

For a quantum strategy:

$$p(a,b|x,y) = \langle \Psi | (E_{xa} \otimes F_{yb}) | \Psi \rangle$$

Marginals

Probability Alices responds w/ a on input x,y:

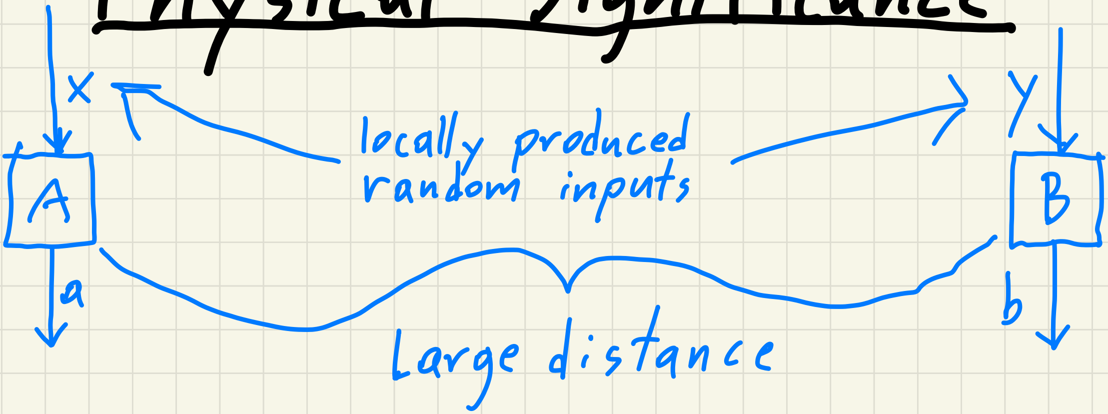
$$\sum_b p(a,b|x,y) = \sum_b \langle \Psi | E_{xa} \otimes F_{yb} | \Psi \rangle = \langle \Psi | E_{xa} \otimes I | \Psi \rangle$$

This does not depend on y!

We say Alice's marginals $p_A(a|x)$ are well-defined and that p is non-signalling (if same holds for Bob).

Exercise: find a non-signalling correlation that wins the CHSH game perfectly (w/ prob 1).

Physical Significance



- Require A + B to respond "quickly"
- Speed of light limit ensures A+B cannot detect each others input
- If A+B consistently win w/ high enough probability, then you can conclude that something nonlocal is occurring, i.e. the behaviour of reality cannot be explained by a local hidden variable theory.

Experimental verification: Aspect et al in a series of experiments.

Homomorphisms

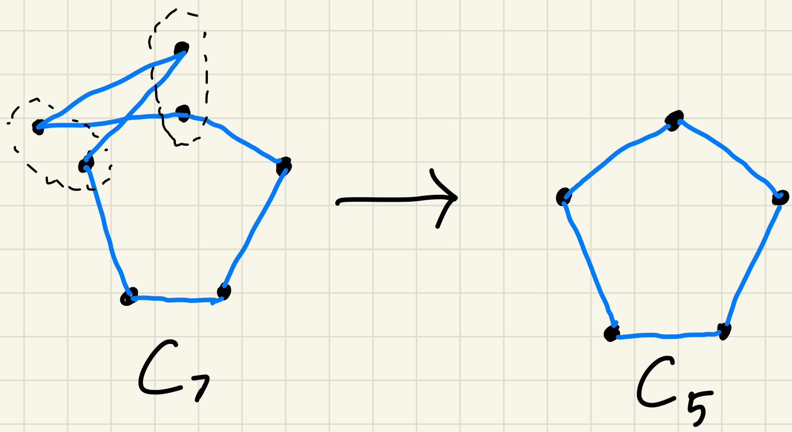
A homomorphism from a graph G to a graph H is a function $f: V(G) \to V(H)$ satisfying

$$u \sim v \Rightarrow f(u) \sim f(v)$$

$f: G \to H$ - f is a homomorphism from G to H

$G \to H$ - there exists a homomorphism from G to H

Examples:



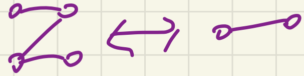
$f: G \to K_n$ is an n-coloring

$$\chi(G) = \min\{n : G \to K_n\}$$

$f: K_n \to G$ is a clique of size n

$$\omega(G) = \max\{n : K_n \to G\}$$

Transitivity: $f_1 : G \to H,\ f_2 : H \to K \Rightarrow f_2 \circ f_1 : G \to K$

Reflexivity: $id : G \to G$ 

Homomorphic equivalence: $G \to H$ + $H \to G$ $=: G \leftrightarrow H$

$\to$ is a pre-order, gives partial order on homomorphic equivalence classes

fibres

H loopless $\Rightarrow f^{-1}(v)$ is an independent set

(G,H)-Homomorphism Game

$I_A = I_B = V(G),\quad O_A = O_B = V(H)$

$$\lambda(h,h' \mid g,g') = \begin{cases} 1 & \text{if } g = g' \Rightarrow h = h' \text{ AND } g \sim g' \Rightarrow h \sim h' \\ 0 & \text{o.w.} \end{cases}$$

$f_A, f_B : V(G) \to V(H)$ – perfect classical deterministic strategy

$f_A(g) = f_B(g)\ \forall g \in V(G)$

$\Rightarrow f_A$ is a homomorphis

Theorem: $G \to H \Leftrightarrow \exists$ perfect classical strategy for (G,H)-hom game

Quantum Strategies

A state $|\psi\rangle$ + POVMS $\mathcal{E}_g$ for $g \in V(G)$ + $\mathcal{F}_{g'}$ for $g' \in V(G)$ give a perfect quantum strategy for the (G,H)-hom game if and only if

$\langle\psi| E_{gh} \otimes F_{g'h'} |\psi\rangle = 0$ whenever ($g \sim g'$ + $h \not\sim h'$) or ($g = g'$ + $h \neq h'$)

If a perfect quantum strategy exists for the (G,H)-hom game, then we say that there is a quantum homomorphism from G to H, and we write $G \xrightarrow{q} H$.

Main Theorem: If $G \xrightarrow{q} H$, then there exists a perfect quantum strategy for the (G,H)-hom game of the following form:

1) $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$
2) $E_{gh} \in \mathbb{C}^{d \times d}$ + $F_{gh} \in \mathbb{C}^{d \times d}$ are projections $\forall g \in V(G), h \in V(H)$
3) $F_{gh} = E_{gh}^T$

Main Corollary: $G \xrightarrow{q} H$ if and only if there exists $d \in \mathbb{N}$ & projections $E_{gh} \in \mathbb{C}^{d \times d}$ for all $g \in V(G)$, $h \in V(H)$ s.t. $E_{gh} E_{g'h'} = 0$ whenever ($g \sim g'$ & $h \not\sim h'$) or ($g = g'$ & $h \neq h'$), and $\sum_{h \in V(H)} E_{gh} = I$.

Proof: First show that if $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$, then for any $E, F \in \mathbb{C}^{d \times d}$ we have $\langle \psi | E \otimes F | \psi \rangle = \frac{1}{d} \mathrm{Tr}(EF^T)$.

($\Rightarrow$): use Alice's operators

($\Leftarrow$): take $|\psi\rangle$ as above and $F_{gh} = E_{gh}^T$ $\forall g \in V(G), h \in V(H)$

Transitivity: $G \xrightarrow{q} H$ & $H \xrightarrow{q} K \Rightarrow G \xrightarrow{q} K$.

Proof: Exercise. $\square$

$g = g' \Rightarrow h = h'$ "function"

<u>Homomorphic Product</u>

$V(G \ltimes H) = V(G) \times V(H)$

$(g,h) \sim (g',h')$ if $(g \sim g'$ & $h \simeq h')$ or $(g = g'$ & $h \neq h')$

Projective packing of G:

$g \mapsto P_g \in \mathbb{C}^{d \times d}$ projection s.t. $g \sim g' \Rightarrow P_g P_{g'} = 0$

Value: $\frac{1}{d} \sum_{g \in V(G)} rk(P_g) = \frac{1}{d} Tr\left(\sum_{g \in V(G)} P_g\right)$

<u>Lemma:</u> $G \not\to H \iff G \ltimes H$ admits a projective packing of value $|V(G)|$.

<u>Proof:</u> Exercise.

Special Cases

Colorings: $E_{gi}E_{g'j}=0$ if $(g\sim g'$ & $i=j)$ or $(g=g'$ & $i\neq j)$

$$G \propto K_n = G \square K_n$$

Quantum chromatic number: $\chi_q(G)=\min\{n : G \xrightarrow{q} K_n\}$

Cliques: $E_{ig}E_{jg'}=0$ if $(i\neq j$ & $g\not\sim g')$ or $(i=j$ & $g\neq g')$

$$K_n \propto G = \overline{K_n \times G}$$

Quantum clique number: $\omega_q(G)=\max\{n : K_n \xrightarrow{q} G\}$